Stabilized finite elements using Lyapunov functions

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ABSTRACT

The standard Galerkin finite element approximation of the convection diffusion equation is known to be numerically unstable for small values of the diffusion parameter [3]. Some authors propose the use of stabilized finite element methods to overcome this problem. Among the most cited methods, we can mention Galerkin least-squares (GLS), streamline upwind Petrov-Galerkin (SUPG) multiscale[2, 4].

In this work, we propose a stabilization for the finite element method using the control theory. Specifically, we formulate the finite element method as a state equation and use the theory Lyapunov functions to propose a feedback control law[1]. The numerical results show that using of Lyapunov functions, we can find a controller that stabilizes the finite element method, even when the diffusion parameter is relatively small. The results encourage and motivate the research in this direction.

Introduction

We consider the 1-D convection diffusion equation and forcing term $f(x,t)$ can be written as [8]:

$$\partial_t w(x,t) = \epsilon \partial_{xx} w(x,t) - \kappa \partial_x w(x,t) + f(x,t), \text{ para } (x,t) \in \Omega \times [0,T), \tag{1}$$

where $(x,\, t) \in \Omega \times [0,T)$, with Dirichlet boundary conditions and inicial conditions, given by:

$$\begin{aligned} w(0,t) &= w(L,t), \\ w(x,0) &= w_0(x); \end{aligned} \tag{2}$$

where $w_0(x) \in L^2(\Omega)$ and $f(x,t) \in L^2(\Omega \times [0,T))$. For simplicity we consider $\Omega = [0,L]$ and $w(0,t) = w(L,t) = 0$; consequently $w(x,t) \in H_0^1(\Omega \times [0,T))$. In this work the forcing term $f(x,t)$ is used to control the equation (1) and is given by $f(x,t) := \sum_{i=1}^{m} b_i(x) u_i(t)$. The parameters $\epsilon,\, \kappa \in \mathbb{R}$ are the coefficients of diffusion and convection, respectively. The equation (1) can be written in matrix form as follows:

$$\begin{cases} M\dot{z} &= Az + Bu(t), \\ z(0) &= z_0 \end{cases} \tag{3}$$

where the matrices $M, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are defined respectively by [5]

$$M = \left[\int_0^L \varphi_i \varphi_j dx\right]_{i,j=1}^m, \; A = -\epsilon \left[\int_0^L \varphi_i' \varphi_j' dx\right]_{i,j=1}^m - \kappa \left[\int_0^L \varphi_i' \varphi_j dx\right]_{i,j=1}^m, \; B = \left[\int_0^L b_i(x) \varphi_j dx\right]_{i,j=1}^m. \tag{4}$$

To solve numerically the equation (3) is necessary to discretize the time variable. To do this we consider the temporal division of the time interval $[0,\, T]$ as

$$0 = t_0 < t_1 < \cdots t_N = T,$$

with equidistant nodes $t_j = j\tau$ and time step $\tau = T/N$. We denote u^j for indicate $u(t_j)$ in the j-th time step. We thus obtain the explicit Euler Discretization of the equation (3):

$$M \frac{z^{j+1} - z^j}{\tau} = Az^j + Bu^j.$$

from which we get

$$\begin{aligned} M z^{j+1} &= (M + \tau A) z^j + \tau B u^j, \quad j = 0, 1, \ldots, N-1, \\ z(0) &= z_0. \end{aligned} \tag{5}$$

Stabilization using Lyapunov

Then we determine the control law for the discretization of the equation system (1) - (2). From the equation (3), we get the standard representation matrix for the design of control:

$$\begin{cases} \dot{z} &= Gz + Nu(t), \\ z(0) &= z_0. \end{cases} \tag{6}$$

where $G = M^{-1}A$ y $N = M^{-1}B$.

To find the controller (or feedback) u^j as a function of z^j we define a function candidate Lyapunov function V given by:

$$V(z^j) = \langle z^j, z^j \rangle.$$

The variation of the Lyapunov function $V(z^k)$ denoted by $\Delta V(z^k)$ in the path given by equation (6) is given by:

$$\begin{aligned} \Delta V(z^k) &= V(z^{k+1}) - V(z^k) \\ &= (z^{k+1})^T P z^{k+1} - (z^k)^T P z^k \\ &= \left[(G + Nu(t))\, z^k\right]^T P (G + Nu(t))\, z^k - (z^k)^T P z^k \\ &= (z^k)^T (G + Nu(t))^T P (G + Nu(t))\, z^k - (z^k)^T P z^k \\ &= (z^k)^T G^T P G z^k + (z^k)^T G^T P N u(t) + (u(t))^T N^T P G z^k \\ &+ (u(t))^T N^T P N u(t) - (z^k)^T P z_k. \end{aligned} \tag{7}$$

So that the variation $\Delta V(z^k)$ means a decaement the control law u^j must have the form (for details see cite [6]):

$$u(t) = -\left[\,(N^T P N + R)^{-1} N^T P N\right] z^k, \tag{8}$$

where $F := (N^T P N + R)^{-1} N^T P G$, $F \in \mathbb{R}^{m\times n}$ and the matrix P satisfies the Riccati equation given by

$$P = G^T P G - G^T P N \left(N^T P N + R\right)^{-1} N^T P N + Q. \tag{9}$$

Numerical results

In this section we present the results of a spatial dimension, primarily for the diffusion equation ($\kappa = 0$), then for the convection diffusion equation, with $\kappa = 1$ and the parameter $\epsilon = 10^{-1}$. Discretize considering $\Omega = [0,1]$ in each example.

Stabilization of the diffusion equation for $\tau = h$ y $\kappa = 0$

Results for one-dimensional diffusion equation:

$$\partial_t w(x,t) = \epsilon \partial_{xx} w(x,t) + f(x,t), \tag{10}$$

In this experiment show the stabilization using Lyapunov functions can relax the condition for τ. For this, we set a value of $\epsilon = 10^{-1}$ and consider $\tau = h$ for the stabilized and unstabilized methods. Reference we also present the results for $\tau = h^2/12\epsilon$ (condition of stability for the Euler method). Following [7, 4], we take the parameter values $q = 10$ and $r = 10^{-1}$ Lyapunov function. The mesh is refined according to the law: $h = h^i = 1/2^i$ for $i = 3, 4, 5$ and 6, respectively.

In the table 1 is presented the spectral radius E. And we see that the explicit Euler-Galerkin method is stable with stability parameter $\tau = h^2/12\epsilon$, while if we take as a parameter $\tau = h$ the method is unstable. This can be seen in the Figures 1 y 2.

In the table 2 we present the spectral radius of matrices E and $E - DF$. It can be seen in the third column of Table 2 in this case the explicit Euler-Galerkin method is stable using the theory of control via Lyapunov functions when we take the parameter $\tau = h$, the results of the experiments we can see in Figures 3 and 4. Note the similarity of the figures.

$h = 1/2^i$	$q = 10, r = 10^{-3}$ y $\epsilon = 10^{-1}$	
	$\tau = h^2/12\epsilon$	$\tau = h$
	uncontrolled $\rho(E)$	uncontrolled $\rho(E)$
$i = 3$	0.8939	8.584
$i = 4$	0.9717	18.6570
$i = 5$	0.9928	38.1240
$i = 6$	0.9982	76.6614

Table 1: Method uncontrolled

$h = 1/2^i$	$q = 10, r = 10^{-3}$ y $\epsilon = 10^{-1}$		
	$\tau = h^2/12\epsilon$	$\tau = h$	$\tau = h$
	uncontrolled $\rho(E)$	uncontrolled $\rho(E)$	controlled $\rho(E - DF)$
$i = 3$	0.8939	8.584	0.8940
$i = 4$	0.9717	18.6570	0.9406
$i = 5$	0.9928	38.1240	0.9694
$i = 6$	0.9982	76.6614	0.9846

Table 2: Method uncontrolled y controlled

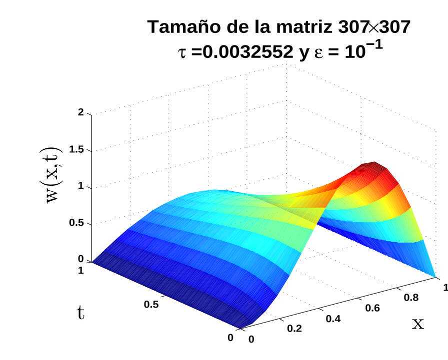

Figure 1: Matrix size 307×307, τ = 0.0032552 y $\epsilon = 10^{-1}$

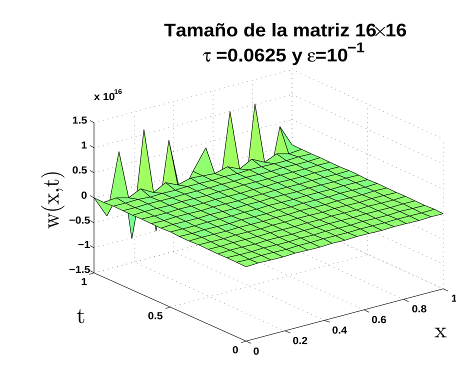

Figure 2: Matrix size 16×16, τ = 0.0625 y $\epsilon = 10^{-1}$

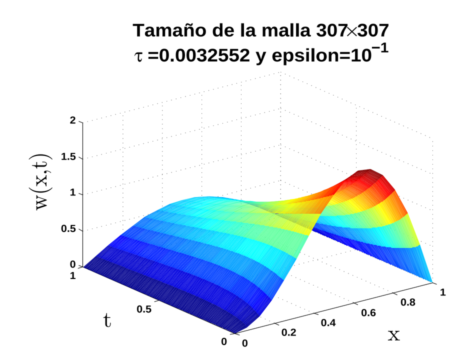

Figure 3: Matrix size 307×307, τ = 0.0032552 y $\epsilon = 10^{-1}$

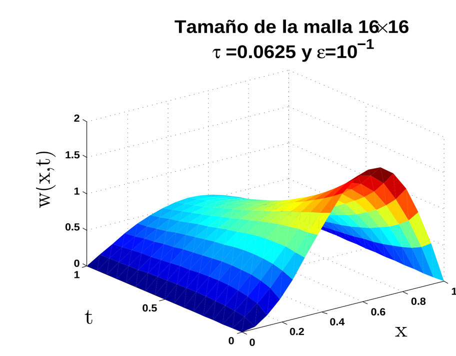

Figure 4: Matrix size 16×16, τ = 0.0625 y $\epsilon = 10^{-1}$

Stabilization of the convection diffusion equation for $\tau = h$ and $\kappa = 1$

We present results for the one-dimensional diffusion convection equation:

$$\partial_t w(x,t) + 1 \cdot \partial_x w(x,t) = \epsilon \partial_{xx} w(x,t) + f(x,t), \tag{11}$$

We perform the numerical experiment considering the case in which the diffusion coefficient is $\epsilon = 10^{-1}$ and $\epsilon = 10^{-3}$. We consider stability as well as parameters $\tau = h$ and $\tau = h^2/12\epsilon$, where h is the size of the mesh, defined by $h = 1/2^i$ and the spatial dimension $n = 1/h$ see table 3 and 4.

Comparing Tables 3 and 4 we see that the explicit Euler-Galerkin controlled $\tau = h$ is stable. However, the explicit Euler-Galerkin method uncontrolled stability with parameter $\tau = h^2/12\epsilon$ is unstable.

$h = 1/2^i$	$q = 1, r = 10^{-2}$ y $\epsilon = 10^{-1}$		
	$\tau = h^2/12\epsilon$	$\tau = h$	$\tau = h$
	no controlado $\rho(E)$	no controlado $\rho(E)$	controlado $\rho(E - DF)$
$i = 3$	0.8627	8.2821	0.6298
$i = 4$	0.9637	18.5025	0.7919
$i = 5$	0.9908	38.0461	0.8923
$i = 6$	0.9977	76.6224	0.9457

Table 3: Method uncontrolled

$h = 1/2^i$	$q = 1, r = 10^{-4}$ y $\epsilon = 10^{-3}$		
	$\tau = h^2/12\epsilon$	$\tau = h$	$\tau = h$
	no controlado $\rho(E)$	no controlado $\rho(E)$	controlado $\rho(E - DF)$
$i = 3$	16.2766	1.5626	0.9999
$i = 4$	8.7923	1.6881	0.9989
$i = 5$	4.4817	1.7210	0.9935
$i = 6$	2.2517	1.7293	0.9816

Table 4: Method uncontrolled y controlled

In the figures show the results we obtained for the Euler method Galerkin explicit-uncontrolled and controlled taking into account the data shown in Tables 3 and 4 with stability parameter $\tau = h$ and $\tau = h^2/12\epsilon$ respectively.

In the figures 5 y 6 see the behavior of the uncontrolled and controlled method when $\epsilon = 10^{-1}$, in both cases the method is stable, taking into account the data in Table 3, for $n = 32$.

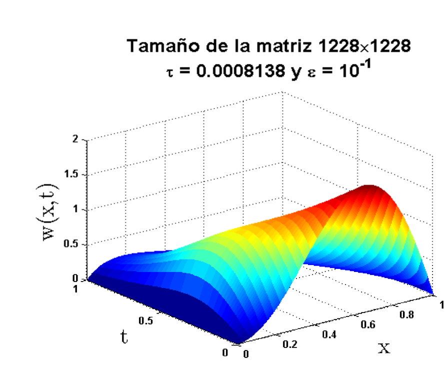

Figure 5: Matrix size 1228×1228,τ = 0.00081380 y $\epsilon = 10^{-1}$

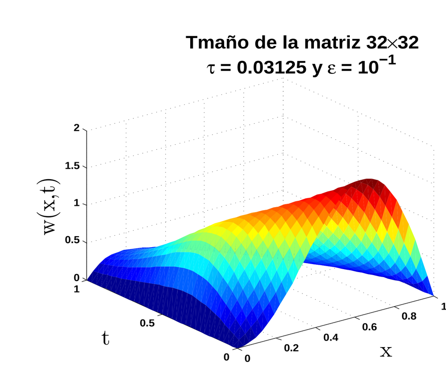

Figure 6: Matrix size 32×32, τ = 0.0312 y $\epsilon = 10^{-1}$

From the results we can infer that the explicit Euler-Galerkin controlled is stable, taking into account the parameter $\tau = h$. In this work we consider the values of q and r parameters of the Lyapunov function, assuming the relationship ($q \gg r$).

Concluding remarks

The use of Lyapunov functions to stabilize the explicit Euler method for both the pure diffusion problem to the problem of convective diffusion.

The stabilization method is independent of the parameter controlled diffusion, within a range experienced, producing good approximations for the explicit *Euler-Galerkin method*.

Finally, numerical experiments to more spatial dimensions than those considered in this work are being implemented, and an analysis of the proposed method.

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